FUNCTIONAL CALCULUS AND POSITIVE-DEFINITE FUNCTIONS

BY

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ABSTRACT. For a LCA group G with dual group $\hat{G}$, let $D(G) = D(\hat{G})$ denote the convex (not closed) hull of $\{\langle x, \gamma \rangle : x \in G, \gamma \in \hat{G}\}$. The set $D(G)$ is the natural domain for functions that operate by composition from the class, $PD_1(\hat{G})$, of Fourier-Stieltjes transforms of probability measures on G to $B(\hat{G})$, the class of all Fourier-Stieltjes transforms on $\hat{G}$. Little is known about the behavior of F on the boundary of $D(G)$. In §1, we show (1) if F operators from $PD_1(G)$ to $B(G)$ and G is compact, then $K(z) = \lim_{t \to 1^-} F(tz)$ exists for all $z \in D(G)$ and K operates from $PD_1(\hat{G})$ to $B(\hat{G})$; (2) if F operates from $PD_1(\hat{G})$ to $PD(\hat{G}) = \bigcup_{r>0} rPD_1(\hat{G})$ and G is compact, then K operates from $PD_1(\hat{G})$ to $PD(\hat{G})$, and so also does $F - K$; (3) if $G = \mathbf{D}_q$, $q \geq 2$, and F operates from $PD_1(\hat{G})$ to $B(\hat{G})$, then $F = K$ on $D(G) \cap \{z: |z| < 1\}$. This third result is shown to be sharp for compact groups of bounded order. In §2, an example is given that fills a gap in the theory of functions operating from $PD_1(\hat{G})$ to $B(\hat{G})$. In §3 we show that most Riesz products and all continuous measures on K-sets have a property that is very useful in proving symbolic calculus theorems. Applications of this are indicated. Some open questions are given in §4.

0. Introduction. We retain the notations of the abstract, and use, in addition, notions and notations of Rudin's monograph [Ru1] without comment. The results of §1 extend previous results concerning functions operating from $PD_1(\hat{G})$ to $PD(\hat{G})$, to give information concerning the behavior of F on the boundary of $D(G)$. Herz [Hz] had observed that this behavior could be complicated, and he and Rider [Ri1] had shown that F has the form $F(z) = \sum a_{mn} z^m \bar{z}^n$ for $z \in \text{Int } D(G)$, where $a_{mn} \geq 0$ for all m, n and $\sum a_{mn} < \infty$ for most groups G.

We thank the referee for many useful comments and suggestions. In particular, the proofs of 1.3 and 3.3 owe much of their simplicity to the referee.

1. Boundary behavior of operating functions.

1.1. *Discussion.* In this section, we consider a complex valued function F,

defined on (a subset of) the unit disc. We suppose that G is an infinite LCA group with dual group $\hat{G}$, and that $F \circ \hat{\mu}$ is a Fourier-Stieltjes transform on $\hat{G}$ whenever μ is a probability measure on G. The question then is: What sort of function is F?

The problem only makes sense when the domain of F is the set $D(G)$, where

(1.1) $D(G) = \cup \{\hat{\mu}(\hat{G}): \mu$ is a probability measure on $G\}$.

It is easy to see that $D(G) = D(\hat{G})$.

A LCA group, G, is *exceptional* [**Ri1**] if G is the product of a finite group with a group of exponent two.

1.2. THEOREM. *Let G be an infinite compact abelian group and let $F: D(G) \to C$ operate (by composition) from $PD_1(\hat{G})$ to $B(\hat{G})$. For $0 \leqslant t < 1$ and $z \in D(G)$, let $K(t,z) = F(tz)$. Then the following hold.*

(i) $\lim_{t \to 1^-} K(t,z) = K(z)$ *exists for all $z \in D(G)$.*

(ii) K *operates from $PD_1(\hat{G})$ to $B(\hat{G})$.*

(iii) *If F operates from $PD_1(\hat{G})$ to $PD(\hat{G})$, then K operates from $PD_1(\hat{G})$ to $PD(\hat{G})$.*

PROOF. If G is not exceptional, then [**Mo**] F has the form $F(z) = \sum_{m,n=0}^{\infty} a_{mn} z^m \bar{z}^n$, $z \in \text{Int} \, D(G)$, where $\sum |a_{mn}| < \infty$. (This result was attempted in [**Gr1**], but there is a gap in the proof.) Hence, (i) and (ii) hold in this case. If F operates from $PD_1^{\cdot}(G)$ to $PD(G)$, then $a_{mn} \geqslant 0$ [**Ri1**] and, therefore, $K \circ \nu = \sum a_{mn} \nu^m \bar{\nu}^n \geqslant 0$ for all probability measures, ν, on G. This establishes the theorem in case that G is not exceptional.

When G is exceptional, [**Mo**] has shown that there exist polynomials $P_0, P_1, \ldots$, homogeneous of degree $0, 1, \ldots$ such that $F(z) = \sum_{k=0}^{\infty} P_k(z)$, for $z \in \text{Int} \, D(G)$ and such that, for all probability measures, ν, on G, $\sum \|P_k \circ \nu\| < \infty$. Now (i) and (ii) follow easily. For (iii), we need to examine Moran's argument more carefully. It follows from the definition of "exceptional" that G is isomorphic to $G \times D_2$.

Let ν be a probability measure on G. Let E be a subset of D_2 that is compact, perfect and of type K_2. We may identify E with the Cantor middle-third set. Let μ be the measure on E that is (identified with) the Cantor-Lebesgue measure, and let $0 < t < 1$ be any dyadic rational. Then it is easy to see that there exist continuous functions g and h on E that take on only the values 1 and -1, are such that $\int g \, d\mu = \int h \, d\mu = t$, and $g + h \geqslant 0$. We may assume, since E is of type K_2, that there are continuous characters λ_1, λ_2 on D_2 such that $g = \lambda_1$ and $h = \lambda_2$ on E.

Since $G \simeq G \times D_2$, $F \circ (\nu \times \mu) \geqslant 0$. Therefore, if $n \geqslant 1$, $c_1, \ldots, c_n \in C$, and $\gamma_1, \ldots, \gamma_n \in \hat{G}$, then

(1.2) $$\int |\sum c_j \gamma_j|^2 \times (\tfrac{1}{2})(\lambda_1 + \lambda_2) \, dF \circ (\nu \times \mu) \geqslant 0.$$

Taking transforms, and using the representation of F in Int $D(G)$, we see that the left-hand side of (1.2) equals

$$\sum_{k=0}^{\infty} \sum_{i,j} c_i \bar{c}_j t^k P_k(\check{\nu}(\gamma_i - \gamma_j)).$$

Since the c's and the γ's were arbitrary, we conclude that $\sum t^k P_k(\check{\nu}) \in PD(\hat{G})$. Since this holds for all dyadic rationals, t, between zero and one, we conclude that $\sum t^k P_k(\check{\nu}) \in PD(\hat{G})$ for all $0 \leqslant t \leqslant 1$. This establishes (iii), and the proof of Theorem 1.2 is complete.

1.3. Theorem. *Let G be an infinite compact abelian group and let $F: D(G) \to C$ operate from $PD_1(\hat{G})$ to $PD(\hat{G})$. Let K be as in 1.2. Then*

$$H(z) = F(z) - K(z), \qquad z \in D(G),$$

defines a function that operates from $PD_1(\hat{G})$ to $PD(\hat{G})$.

Remark. Even if G is the circle group, $\mathbf{T}$, the function H may be quite complicated. It must have, as Herz [**Hz**] observed, the property that, on each singly generated subgroup of the unit circumference (the boundary of $D(\mathbf{T})$), K agrees with a positive-definite function. There is no guarantee that K equals a positive-definite function on all doubly generated subgroups. An example appears in [**Hz**].

1.4 Lemma. *Let G be a compact, infinite abelian group, and $F: D(G) \to \mathbf{C}$.*

(i) *F operates from $PD_1(\hat{G})$ to $PD(\hat{G})$ if and only if F operates from $PD_1(\Lambda)$ to $PD(\Lambda)$ for all finitely generated subgroups, Λ, of $\hat{G}$.*

(ii) *Suppose that $G = \Phi \times \mathbf{T}^k$, where Φ is a finite abelian group. Then F operates from $PD_1(\Phi \times \mathbf{Z}^k)$ to $PD(\Phi \times \mathbf{Z}^k)$ if and only if F operates from $PD_1(\Phi \times \mathbf{Q}^k)$ to $PD(\Phi \times \mathbf{Q}^k)$, where $\mathbf{Q}$ denotes the rational numbers.*

(iii) *Suppose that Λ is a subgroup of $\hat{G}$ of infinite index. Let $\omega, \nu \in M(G)$ and suppose that ν is discrete. If $\hat{\omega} = \hat{\nu}$ on $\hat{G} \backslash \Lambda$, then ν is the discrete part of ω.*

Proof. (i) Immediate from the equality $\|F \circ \nu\| = F(1)$.

(ii) This follows from (i) and the fact that finitely generated subgroups of $\mathbf{Q}$ are isomorphic to $\mathbf{Z}$.

(iii) Let $L = \{x \in G: \langle x, \lambda \rangle = 1, \lambda \in \Lambda\}$. Let ρ be Haar measure on L, normalized to have norm one. Then ρ is a continuous measure. The conclusion follows from consideration of the Fourier-Stieltjes transform of $\nu + \rho * (\omega - \nu)$.

This completes the proof of Lemma 1.4.

1.5. Proof of Theorem 1.3. In view of 1.4(i), it will be sufficient to prove

that if Λ is a finitely generated subgroup of $\hat{G}$, and if F operates from $PD_1(\hat{G})$ to $PD(\hat{G})$, then H operates from $PD_1(\Lambda)$ to $PD(\Lambda)$.

We claim that it will be sufficient to prove this when Λ has infinite index in $\hat{G}$. For, suppose that whenever Λ is a finitely generated subgroup of infinite index in the discrete abelian group $\hat{G}$, and $F: D(G) \to \mathbf{C}$ operates from $PD_1(\hat{G})$ to $PD(\hat{G})$, then H operates from $PD_1(\Lambda)$ to $PD(\Lambda)$. Let $\hat{G}$ be an infinite discrete abelian group and let Λ be a finitely generated subgroup of $\hat{G}$. We must show that H operates from $PD_1(\Lambda)$ to $PD(\Lambda)$ whenever F operates from $PD_1(\hat{G})$ to $PD(\hat{G})$. When Λ has infinite index, we have assumed that this holds. When Λ has finite index, then Λ is an infinite, finitely generated abelian group, and therefore Λ has the form $\Phi \times \mathbf{Z}^k$, where Φ is a finite abelian group and $k \geqslant 1$. By 1.4(ii), F operates from $PD_1(\Phi \times \mathbf{Q}^k)$ to $PD(\Phi \times \mathbf{Q}^k)$. In the larger group, Λ has infinite index, so H (by assumption) operates from $PD_1(\Lambda)$ to $PD(\Lambda)$. This establishes the claim.

Let, therefore, Λ be a subgroup of G having infinite index and being finitely generated. (The finitely generated assumption is now superfluous.) Suppose that $\hat{\mu} \in PD_1(\hat{G})$. Then, the extension of $\hat{\mu}$ to all of G, given by setting $\hat{\mu} = 0$ on $\hat{G} \backslash \Lambda$, is an element of $PD_1(\hat{G})$. (See [Gr1] for a proof.) For convenience, let us assume that $\hat{G}$ is countable. By 1.4(i), there will be no loss of generality in this.

Let ν be any discrete probability measure on G whose support is a dense subset of the support of μ. Suppose that $\gamma \in \hat{G}$ and $\hat{\mu}(\gamma) \in \partial D(G)$. Then either $|\hat{\mu}(\gamma)| = 1$, in which case γ is constant a.e. $d\mu$ (and a.e. $d\nu$) or $|\hat{\mu}(\gamma)| < 1$, in which case γ takes on values on the support of μ that are exactly two adjacent vertices of the polygon $D(G)$. In this case, it follows, as in the $|\hat{\mu}(\gamma)| = 1$ case, that $\hat{\nu}(\gamma) \in \partial D(G)$ also. It is now easy to see that $\hat{\mu}(\gamma) \in \partial D(G)$ if and only if $\hat{\nu}(\gamma) \in \partial D(G)$.

Now, $K(\hat{\nu})$ is the transform of a discrete measure (see the representation of $K(\hat{\nu})$ used in the proof of 1.2), and $H(\hat{\nu})$ vanishes off of Λ, since, for $\gamma \notin \Lambda$, $\hat{\mu}(\gamma)$ and $\hat{\nu}(\gamma)$ both belong to Int $D(G)$. By 1.4(iii), $K \circ \nu$ is the discrete part of $F \circ \nu$. But $F \circ \nu \geqslant 0$ (by assumption) and therefore its continuous part, $H \circ \nu$, is a nonnegative measure. We have proved the theorem insofar as transforms of discrete measures are concerned. The general case follows on approximating μ by a net of discrete probability measures, $\{\nu_\alpha\}$. The ν_α can be chosen in such a manner that, for each finite subset, E, of $\hat{G}$, there exists $\alpha(E)$ such that for each $\alpha \geqslant \alpha(E)$, if $\gamma \in E$ and $\hat{\mu}(\gamma) \in \partial D(G)$, then $\hat{\nu}_\alpha(\gamma) = \hat{\mu}(\gamma)$. It will then follow that $\{F \circ \nu_\alpha\}$ will converge weak* to $F \circ \mu$. Since $\{K \circ \nu_\alpha\}$ will converge to $K \circ \mu$ (by the representations of K that are used in the proof of 1.2), we conclude that the positive measures $H \circ \nu_\alpha$ converge weak* to the (now necessarily) nonnegative measure $H \circ \mu$. The proof of 1.3 is complete.

1.6. We now turn to the case of groups with finite exponent (bounded

order). In this case, $D(G)$ is a polygon, and it is natural to ask what kind of behavior F has on the faces of $D(G)$. We have two results. Theorem 1.6, for G (a power of) $\mathbf{D}_q$, shows that H vanishes except at the vertices of $D(G)$. When G is not (a power of) $\mathbf{D}_q$, then F may be very badly behaved on $D(G)$, as Theorem 1.7 shows. We retain the notation (F, K, H) of the preceding paragraphs.

THEOREM. *Let $G = \mathbf{D}_q$, for $q \geqslant 3$, and let $F: D(G) \to \mathbf{C}$ operate from $PD_1(\hat{G})$ to $B(\hat{G})$. Then $H(z) = 0$ for $z \in D(G)$, $|z| < 1$.*

PROOF. If $z \in \operatorname{Int} D(G)$, the conclusion is immediate. Otherwise, $z = \alpha r^j + \beta r^{j+1}$, where $0 < \alpha, \beta$; $\alpha + \beta = 1$; and r is a primitive qth root of unity. Let x be a generator of $\mathbf{Z}_q$ (the group of the qth roots of unity, and let ν be the measure on $\mathbf{Z}_q$ given by $\nu = \alpha \delta_{jx} + \beta \delta_{(j+1)x}$. Let μ be the measure on $\mathbf{D}_q$ that is the infinite product of ν with itself (infinitely often). Let $(\gamma_1, \gamma_2, \dots) \in \hat{\mathbf{D}}_q$. Then

$$\hat{\mu}(\gamma_1, \gamma_2, \dots) = \prod_{k=1}^{\infty} [\langle jx, \gamma_k \rangle \alpha + \langle (j+1)x, \gamma_k \rangle \beta].$$

If $1 \leqslant k(1) \neq k(2) < \infty$ and $\gamma_{k(1)} \neq 0$, and $\gamma_{k(2)} \neq 0$, then $\hat{\mu}(\gamma_1, \gamma_2, \dots) \in \operatorname{Int} D(G)$. (This follows from a straightforward calculation using the sum formulae for sine and cosine.) Therefore

(1.3) $\{\gamma: \hat{\mu}(\gamma) \in \partial D(\hat{D}_q)\} = \{(\gamma_1, \gamma_2, \dots): \gamma_j \neq 0 \text{ for at most one } j\}.$

The set on the right-hand side of (1.3) is a union of q independent sets with $\{0\}$, and is therefore [**Ru1**, 5.7.5] a Sidon set. (There is no need to use Drury's Theorem here.) Now, the support of $H(\hat{\mu})$ is a subset of the set in (1.3). By [**Ru1**, 5.7.7], $H(\hat{\mu}) \in L^1(\hat{\mathbf{D}}_q)$. But if $H(z) \neq 0$, then $H(\hat{\mu})$ takes on the value $H(z)$ infinitely often. This contradiction proves that $H(z) = 0$ for all $z \in \partial D(\mathbf{D}_q)$, $|z| < 1$.

1.7. THEOREM. *Let Φ be a finite group of cardinality n and let $q \geqslant 2$ be an integer that strictly divides the exponent of Φ. Let $G = \Phi \times \mathbf{D}_q$. Let $F: D(G) \to \mathbf{C}$ be such that $F = 0$ on $\operatorname{Int} D(G)$ and $F(\bar{z}) = \overline{F(z)}$ for all $z \in D(G)$. Suppose that*

(1.4) $F(1) > (nq^{q^n})^2 \sup\{|F(z)|: z \in D(G), z \neq 1\}.$

Then F operates from $PD_1(\hat{G})$ to $PD(\hat{G})$.

We shall prove 1.7 by reducing the argument to one concerning finite groups.

1.8. LEMMA. *Let Φ, n, q, and G be as in 1.7. Let $F: D(G) \to \mathbf{C}$ vanish on*

Int $D(G)$. *Then F operates from $PD_1(\hat{G})$ to $PD(\hat{G})$ if and only if F operates from $PD_1(\Phi \times (\mathbf{Z}_q)^{q^n})$ to $PD(\Phi \times (\mathbf{Z}_q)^{q^n})$.*

1.9. PROOF OF THEOREM 1.7, ASSUMING LEMMA 1.8. It will be sufficient to show that if F satisfies the hypotheses of 1.7, then $F \circ \hat{\mu} \in PD(\Phi \times (\hat{\mathbf{Z}}_q)^{q^n})$ whenever $\hat{\mu} \in PD_1(\hat{\Phi} \times (\hat{\mathbf{Z}}_q)^{q^n})$. For convenience, let $L = \Phi \times (\mathbf{Z}_q)^{q^n}$. Let $\Lambda = \{\lambda \in \hat{L}: \hat{\mu}(\lambda) = 1\}$. Then $\hat{\mu}$ and $F \circ \hat{\mu}$ are constant on cosets of Λ. Therefore [**Ru1**, 2.7], [**Gr1**], $F \circ \hat{\mu} \in PD(\hat{L})$ if and only if the function defined by $F \circ \hat{\mu}$ on $\hat{L}/\Lambda$ belongs to $PD(\hat{L}/\Lambda)$. The cardinal, m, of $(\hat{L}/\Lambda)^{\hat{}}$ is at most nq^{q^n}, of course. For each $x \in (\hat{L}/\Lambda)^{\hat{}}$,

$$(1.5) \qquad F \circ \mu(\{x\}) = \sum \{F(\hat{\mu}(\gamma))\langle x, \gamma \rangle : \gamma \in \hat{L}/\Lambda\}(m)^{-1}.$$

Because $\hat{\mu}(-\gamma) = \hat{\mu}(\gamma)^-$ and $F(\bar{z}) = F(z)^-$ for $z \in D(G)$, the right-hand side of (1.5) is real. Furthermore, for $x \in \{x \in L: \langle x, \lambda \rangle = 1, \lambda \in \Lambda\} = (\hat{L}/\Lambda)^{\hat{}}$,

$$(1.6) \qquad \left| \sum \{F(\hat{\mu}(\gamma))\langle x, \gamma \rangle : \gamma \in \hat{L}/\Lambda, \gamma \neq 0\} \right| \leqslant nq^{q^n}S,$$

where $S = \sup\{|F(z)|: z \in D(G), z \neq 1\}$. Now, (1.4)–(1.6) imply that $F \circ \mu(\{x\}) \geqslant 0$ for all $x \in (\hat{L}/\Lambda)^{\hat{}}$. This completes the proof that 1.8 implies 1.7.

1.10. PROOF OF LEMMA 1.8. Let p denote the (common) exponent of G and Φ. We have a number of steps. Most of the details are simple and are omitted.

(A). If $z \in D(\mathbf{D}_q)$ and $|z| < 1$, then $z \in \text{Int} D(G)$. (This is a simple geometric argument. A sketch of the p- and q-gons involved should be persuasive.)

(B). Every probability measure, μ, on G has the form $\mu = \sum_{x \in \Phi} \delta_x \times \mu_x$, where $\mu_x \in M(\mathbf{D}_q)$, $\mu_x \geqslant 0$ for all $x \in \Phi$, and $\sum_{x \in \Phi} \|\mu_x\| = 1$.

(C). If $(\gamma, \rho) \in \hat{\Phi} \times \hat{\mathbf{D}}_q$ and if, for some $x \in \Phi$, $|\hat{\mu}_x(\rho)| < \|\mu_x\|$, then $\hat{\mu}(\gamma, \rho) \in \text{Int} D(G)$.

(D.) Let $\Lambda_1 = \{\lambda \in \hat{\mathbf{D}}_q: |\hat{\mu}_x(\lambda)| = \|\mu_x\|$ for all $x \in \Phi\}$. Then Λ_1 is a subgroup of $\hat{\mathbf{D}}_q$ and $F \circ \hat{\mu}$ is supported on $\hat{\Phi} \times \Lambda_1$. We may assume that $\hat{\mu}$ is also supported on $\hat{\Phi} \times \Lambda_1$. Set $\Lambda = \{\lambda \in \hat{D}_q: \sum \hat{\mu}_x(\lambda) = 1\}$.

(E). Then $\text{Card}(\Lambda_1/\Lambda) \leqslant q^n$. (Because, for each $x \in \Phi$, $\text{Card}\{\hat{\mu}_x(\lambda): \lambda \in \Lambda_1\} \leqslant q$.)

(F). Since $\hat{\mu}$ and $F \circ \hat{\mu}$ are constant on cosets of Λ, it is enough to prove that $F \circ \hat{\mu}$ belongs to $PD(\hat{\Phi} \times (\Lambda_1/\Lambda))$. But $\text{Card}(\Lambda_1/\Lambda) \leqslant q^n$ implies that Λ_1/Λ is a quotient of $(\hat{\mathbf{Z}}_q)^{q^n}$. (The exponent may be too large here.)

(G). We thus lift $\hat{\mu}$ and $F \circ \hat{\mu}$ to functions on $\hat{\Phi} \times (\hat{\mathbf{Z}}_q)^{q^n}$ and apply the hypotheses. We conclude that $F \circ \hat{\mu} \in PD(\hat{\Phi} \times (\hat{\mathbf{Z}}_q)q^n)$, and, therefore, that $F \circ \hat{\mu} \in PD(\Lambda_1/\Lambda)$.

We have proved one direction of the lemma. The other direction is obvious.

2. An example. The following result is due to Moran [**MO**].

2.1. THEOREM. *Let G be a LCA group. Suppose that either* (i) *G contains a perfect Kronecker set or a perfect set of type K_p, for $p \geqslant 3$, or that* (ii) *G contains an infinite compact open subgroup of exponent two and G/H has an element of infinite order. Suppose that $F\colon D(G) \to \mathbf{C}$ operates from $PD_1(\hat{G})$ to $B(\hat{G})$. Then there exist numbers a_{mn} $(m, n = 0, 1, 2, \ldots)$ such that*

$$(2.1) \qquad F(z) = \sum_{m,n=0}^{\infty} a_{mn} z^m \bar{z}^n \quad \text{for } z \in \operatorname{Int} D(G),$$

and

$$(2.2) \qquad \sum_{m,n=0}^{\infty} |a_{mn}| < \infty.$$

When G has an open compact subgroup H of exponent two and G/H has finite exponent, then [**Mo**] (2.1)–(2.2) will fail for some functions operating from $PD_1(G)$ to $B(G)$. The next theorem takes care of the missing case: G/H is a torsion group of infinite exponent (exponent zero).

2.2. THEOREM. *Let G be a nondiscrete LCA group. Let H be an open subgroup of G of exponent two. Suppose that G/H is a torsion group. Let β be any complex number of modulus one that is not a root of unity. Then $F(z) = (2 + \beta z + \bar{z})^{-1}$ is defined in $D(G)$, operates from $PD_1(\hat{G})$ to $B(\hat{G})$, and (2.2) fails for F. The coefficients a_{mn} $(m, n = 0, 1, 2, \ldots)$ are uniquely defined and (2.1) holds with convergence uniform on compact subsets of $\operatorname{Int} D(G)$.*

PROOF. Let $H(w, z) = (2 + \beta w + \bar{z})^{-1}$. Then H is holomorphic in the two complex variables w, z, in the region $|w| < 1, |z| < 1$ and $H(w, z) = \sum a_{mn} w^m z^n$ with uniform convergence on compact subsets of that region. Since H has a pole on $|w| = 1, |z| = 1$, (2.2) fails. The set $\{w = \bar{z}\}$ is a set of uniqueness for the holomorphic functions in the two dimensional disc, and this shows that if $F(z) = \sum b_{mn} z^m \bar{z}^n$ holds in a neighborhood of zero (with absolute convergence there), then $a_{mn} = b_{mn}$ for all m, n. (That F does have such a representation is obvious because F is real-analytic in $\operatorname{Int} D(G)$.) Finally, the poles of H miss $\{(w, z)\colon w = \bar{z}, w \in D(G)\}$, and therefore H and F extend continuously to all of $D(G)$.

We now have only to prove that F operates as required.

Since H is an open subgroup of exponent two, every probability measure, μ, on G has the form $\sum_{j=1}^{\infty} \alpha_j \delta_{x_j} * \nu_j$ where $\nu_j \in M(H)$ are probability measures, $\sum \alpha_j = 1$, all $\alpha_j \geqslant 0$, and the sets $x_j + H$ are distinct cosets of H. (This representation is by no means unique, in general.) Since H has exponent two, $\tilde{\nu}_j = \nu_j$ for all j.

Because G/H has no elements of infinite order, we may assume that x_1 has order $p \geqslant 2$, and $\alpha_1 > 0$. (If all x_j have order 1, then $\mu = \delta_0$ and $F \circ \hat{\mu}$ $= F(1)\hat{\delta}_0$ which certainly belongs to $B(\hat{G})$.) Now, $F(z)$ has the power series expansion

$$F(\bar{z}) = \frac{1}{2} \sum_{k=1}^{\infty} 2^{-k}(\beta z + \bar{z})^k, \quad \text{for } |z| < 1.$$

We shall estimate $\|(\beta\mu + \hat{\mu})^k\|$ for an appropriate integer k. Because β is not a root of unity, β has infinite order in $\mathbf{T}$, and therefore there exists an integer, $n \geqslant 1$, such that

(2.1) $|1 + \beta^{np}| < \frac{1}{2}.$

Then

$$\|(\beta\mu + \tilde{\mu})^{np}\| = \|[\textstyle\sum (\alpha_j(\beta\delta(x_j) + \delta(-x_j)))]^{np}\|$$

$$= \|\alpha_1^{np}(1 + \beta^{np})\delta(npx_1)\nu_1^{np} + \omega\|$$

where $\|\omega\| \leqslant 2^{np} - 2\alpha_1^{np}$.

Thus,

$$\|(\beta\mu + \tilde{\mu})^{np}\| \leqslant 2^{np} - 2\alpha_1^{np} + \tfrac{1}{2}\alpha_1^{np} < 2^{np} - \alpha_1^{np}.$$

Therefore

$$\|F \circ \hat{\mu}\| \leqslant \frac{1}{2} \sum_{1}^{\infty} 2^{-k}\|(\beta\mu + \tilde{\mu})^k\|$$

$$\leqslant \frac{1}{2}\left(\sum_{j=1}^{np} 2^{-j}\|(\beta\mu + \tilde{\mu})^j\|\right) \sum_{k=1}^{\infty} 2^{-knp}\|(\beta\mu + \tilde{\mu})^{knp}\|$$

$$\leqslant \frac{1}{2} \sum_{j=1}^{np} 2^{-j}\|(\beta\mu + \tilde{\mu})^j\| \sum_{k=1}^{\infty} 2^{-knp}(2^{np} - \alpha_1^{np})^k$$

$$< \infty.$$

So $F \circ \hat{\mu} \in B(\hat{G})$.

This completes the proof of 2.2.

3. **Two useful properties of a measure.** Suppose that μ is a regular Borel measure of norm one on the LCA group G, and that the dual group of G is $\hat{G}$. For numbers $\varepsilon > 0$, $K > 0$, an integer $n \geqslant 1$, and a subset E of $\hat{G}$, consider the following:

(3.1) $\|\hat{\mu}^{n+1}|_E\|_{B(E)} < \varepsilon$;

(3.2) $|c_j| \leqslant K \ (1 \leqslant j \leqslant n)$ imply

$$\left\|\sum_{j=1}^{n} c_j(\hat{\mu}^j)|_E\right\|_{B(E)} \geqslant \sum_{j=1}^{n} |c_j| - \varepsilon;$$

(3.3) $\|(\hat{\mu}^r \bar{\hat{\mu}}^s)|_E\|_{B(E)} < \varepsilon$ if $r + s \geqslant n + 1$; and

(3.4) $|a_{rs}| \leqslant K \ (0 \leqslant r, s \leqslant n, r + s \leqslant n)$ imply

$$\left\|\sum_{r+s \leqslant n} a_{rs}(\hat{\mu}^r \bar{\hat{\mu}}^s)|_E\right\|_{B(E)} \geqslant \sum |a_{rs}| - \varepsilon.$$

3.1. DEFINITION. If for every $\varepsilon > 0$, $K > 0$, and integer $n \geqslant 1$, there exists a finite subset E of $\hat{G}$ such that (3.1) and (3.2) hold, we shall say that μ has *property* P. If for every $\varepsilon > 0$, $K > 0$ and integer $n \geqslant 1$, there exists a finite subset E of $\hat{G}$ such that (3.3) and (3.4) hold, we shall say that μ has *property* $\tilde{P}$.

This section is organized as follows. We state three theorems that establish the existence of measures with properties P on all nondiscrete LCA groups and measures with property $\tilde{P}$ on many LCA groups (Theorems 3.2–3.4). (The second and third results are proved at once, while the proof of 3.2 is put off until the very end of the section.) We then indicate applications of these existence theorems to the symbolic calculus of the measure algebra and one new result is given in that paragraph, 3.7. The terminology used in the statement of 3.2 is defined in paragraph 3.8.

3.2. THEOREM. *Let v be a Riesz product on the compact abelian group, G, that is based on Θ and* **a**.

(i) *Suppose that*

(3.5) $$\limsup\{|\mathbf{a}(\theta)|: \theta + \theta = 0, \theta \in \Theta\} < 2,$$

and

(3.6) $$\sum_{\theta \in \Theta} |\mathbf{a}(\theta)|^{2n} = \infty \quad \text{for } n = 1, 2, \ldots.$$

Then v has property P.

(ii) *Suppose that*

(3.7) $$\sum \{|\mathbf{a}(\theta)|^{2n}: \theta \in \Theta, \theta + \theta \neq 0\} = \infty \quad \text{for } n > 1, 2, \ldots,$$

and

(3.8) $$\mathbf{a}(\theta) = r_\theta e^{ix} \text{ for } \theta \in \Theta, \theta \vdash \theta \neq 0, 0 \leqslant r_\theta \leqslant \tfrac{1}{2},$$

and e^{ix} has infinite order in T.

Then v has property $\tilde{P}$.

COROLLARY. *Every nondiscrete LCA group supported probability measures with property* P.

PROOF OF COROLLARY. Every infinite compact abelian group supports Riesz products that satisfy (3.5)–(3.6) [HeZu], [BM1]. The routine lifting from **T** to **R**, combined with the structure theorem for LCA groups, now yields the proof.

3.3. THEOREM. *Let G be a nondiscrete LCA group and let K be a perfect set of type K_p ($p \geqslant 2$) or a Kronecker set. Let μ be a continuous probability measure concentrated on K.*

(i) *Then $\frac{1}{2}(\mu + \tilde{\mu})$ has property* P.

(ii) *If K is a Kronecker set or of type K_p for $p \geqslant 3$, then μ has property $\tilde{\mathrm{P}}$.*

PROOF. Suppose that K is a Kronecker set. Let ν be a Riesz product on **T** that has property $\tilde{\mathrm{P}}$. Then it is easy to see that there exists a sequence $\{\gamma_n\} \subseteq \hat{G}$ of characters such that the maps $\{\gamma_n^*\}$ induced from $M(G)$ to $M(\mathbf{T})$ have $\nu = \lim \gamma_n^* \mu$ and $\frac{1}{2}(\nu + \tilde{\nu}) = \lim \gamma_n^* \frac{1}{2}(\mu + \tilde{\mu})$. (These limits are weak*.) It is now easy to verify that the P and $\tilde{\mathrm{P}}$ properties can be lifted to μ and $\frac{1}{2}(\mu + \tilde{\mu})$.

If K is a set of type K_p, a similar argument, with $\mathbf{D}_p$ in place of **T**, applies.

We now prove the existence of measures with property $\tilde{\mathrm{P}}$ on noncompact groups. The lifting from **T** to **R** will preserve property $\tilde{\mathrm{P}}$ (as indicated already), so we need deal only with groups having a compact open subgroup. A slightly more general statement can be given, however.

3.4. THEOREM. *Let G be a LCA group, and let H be an open subgroup of G. Suppose that μ is a measure on H that has property* P, *and that $x \in G$ is such that $x + H$ has infinite order in G/H. Then $\delta_x * \mu$ has property $\tilde{\mathrm{P}}$.*

It follows from 3.2–3.4 that the only nondiscrete groups. G, that may not have measures with property $\tilde{\mathrm{P}}$ are those having an open compact subgroup, H, of exponent two such that G/H is a torsion group. It follows from 2.2, and our applications (paragraph 3.7) that if G/H is a torsion group and H has exponent two, then G supports no measures wtih property P.

We now proceed with the proof of 3.4. The reduction to the case that $G = \mathbf{Z} \times H$ and $x = (1,0)$ is routine and is omitted.

3.5. LEMMA. *Let H_1 and H_2 be LCA groups with H_1 discrete, $F \subseteq \hat{H_2}$, $x_1, \ldots, x_n \in H_1$, and $\nu_1, \ldots, \nu_n \in M(H_2)$. Then*

$$(3.9) \qquad \left\| \left(\sum_{j=1}^{n} \delta_{x_j} \times \nu_j \right)^{\hat{}} \bigg|_{\hat{H_1} \times F} \right\|_{B(\hat{H_1} \times F)} \geqslant \sum_{j=1}^{n} \|\hat{\nu_j}|_F\|_{B(F)},$$

provided that the x_j are distinct.

PROOF. Any element of $M(H_1 \times H_2)$ has the form

(3.10) $\quad \omega = \sum \delta_{y_j} \times \omega_j,$ where $y_j \in H_1, \nu_j \in M(H_2)$ for $1 \leqslant j < \infty,$

and the y_j are distinct and unique. The norm of ω is given by $\sum \|\omega_j\|.$ Let us choose any $\omega \in M(H_1 \times H_2)$ with $\hat{\omega}|_{\hat{H}_1 \times F} = (\sum \delta_{x_j} \times \nu_j)\check{}|_{\hat{H}_1 \times F},$ and write ω in the form (3.10). Then

(3.11) $\quad \sum \langle y_i, \gamma \rangle \hat{\omega}_i(\rho) = \sum \langle x_j, \gamma \rangle \hat{\nu}_j(\rho),$ for all $(\gamma, \rho) \in \hat{H}_1 \times F.$

For each fixed $\rho \in F,$ the uniqueness of the Fourier-Stieltjes transform (from $M(H_1)$ to $C(\hat{H}_1)$) implies that

$$\hat{\omega}_i(\rho) = \hat{\nu}_j(\rho) \qquad \text{if } y_i = x_j \text{ for some } i, j,$$
$$\hat{\omega}_i(\rho) = 0 \qquad \text{if } x_i \neq y_j \text{ for all } i \text{ and } j.$$

Upon renumbering the elements, we may conclude that

$$\hat{\omega}(\gamma, \rho) = \sum_{j=1}^{n} \langle x_j, \gamma \rangle \hat{\omega}_j(\rho) = \sum_{j=1}^{n} \langle x_j, \gamma \rangle \hat{\nu}_j(\rho)$$

for all $(\gamma, \rho) \in \hat{H}_1 \times F.$ It now follows that $\hat{\omega}_j(\rho) = \hat{\nu}_j(\rho)$ for all $\rho \in F$ and $1 \leqslant j \leqslant n.$ Therefore, (3.9) follows from the norm computation for $\omega.$

    3.6. PROOF OF THEOREM 3.4. We fix $n \geqslant 1, \varepsilon > 0, K > 0$ and choose a compact set $F \subseteq \hat{H}$ such that if $c_1, \ldots, c_{2n} \in \mathbb{C}$ and $|c_1| \leqslant 4nK, \ldots, |c_{2n}| \leqslant 4nK,$ then

(3.12) $$\left\| \sum c_j \hat{\mu}^j \Big|_F \right\|_{A(F)} \geqslant \sum |c_j| - \frac{\varepsilon}{2n},$$

and such that $\|\hat{\mu}^{n+1}|_F\|_{A(F)} < \varepsilon.$ We set $E = \mathbf{T} \times F.$ Let $W = \{(r,s): 0 \leqslant r,s; 0 \leqslant r + s \leqslant n\},$ and let $a_{rs}, (r,s) \in W,$ be complex numbers of modulus at most $K.$ Then Lemma 3.5 implies that

(3.13)
$$\left\| \left(\sum a_{rs} \delta_{(r-s)x} * \mu^{r+s} \right)\check{} \Big|_E \right\|_{A(E)}$$
$$\geqslant \sum_{m=-n}^{n} \left\| \sum_{\substack{r-s=m \\ (r,s) \in W}} a_{rs} \hat{\mu}^{r+s} \Big|_F \right\|_{A(F)}.$$

An application of 3.12 now yields (3.4). We now choose a finite set, $E' \subseteq E$ such that (3.13) holds. As for (3.3), we note that it follows from the fact that translation is an isometry of $M(G),$ so

$$\|(\delta_{kx} * \hat{\mu}^{n+1})\check{}|_E\|_{A(E)} = \|\hat{\mu}^{n+1}|_F\|_{A(F)}.$$

Theorem 3.4 is proved.

3.7. *Applications.* All of the global symbolic calculus theorems [**Ru1**], [**Ri1**], [**Hz**], [**Mo**], [**V1**] follow from the existence of measures with properties P and $\tilde{\mathrm{P}}$. Here is a new result, that does not seem to be obtainable by previous methods. The proof is a typical application of measures with property $\tilde{\mathrm{P}}$.

THEOREM. *Let G be a compact abelian group that is not exceptional. Let $F: D(G) \to \mathbf{C}$ operator from $PD_1(\hat{G}) \cap C_0(\hat{G})$ to $PD(\hat{G})$. Then $F(z) = \sum a_{mn} z^m \bar{z}^n$ for z in a neighborhood of 0, where all $a_{mn} \geqslant 0$ and $\sum a_{mn} < \infty$.*

PROOF. As in [**Ri1**], [**Hz**], [**Mo**], we may assume that F is real-analytic in a neighborhood of 0, so (see, e.g. [**Gr1**]) $F(z) = \sum a_{mn} z^m \bar{z}^n$ for z in that neighborhood, say for $|z| < \delta$, and $\sum |a_{mn}| \delta^{m+n} < \infty$. Let μ be a Riesz product on G, with property $\tilde{\mathrm{P}}$, that has $|\hat{\mu}(\gamma)| < \delta$ if $\gamma \neq 0$, and $\hat{\mu} \in C_0(G)$. It follows from (3.3)–(3.4) and the representation of F in $|z| < \delta$ that $\sum |a_{mn}| < \infty$. Furthermore, examination of transforms shows that

$$F \circ \mu = \sum a_{mn} \mu^m * \bar{\mu}^n + (F(1) - \sum a_{mn})h,$$

where h is Haar measure on G. All of the measures $(\mu^m * \bar{\mu}^n, h)$ that appear on the right-hand side of the above representation of $F \circ \mu$ are mutually pairwise singular. Since $F \circ \mu$ is nonnegative, we must have $a_{mn} \geqslant 0$.

This completes the proof of the theorem.

REMARK. The theorem above is the common generalization of the theorems of Varopoulos on the one hand, and Herz and Rider on the other. See [**V1**], [**Hz**], [**Ri1**].

3.8. *Preliminaries concerning Riesz products.* Let G be a compact abelian group with dual group $\hat{G}$. A subset, Θ, of G is *dissociate* if every $\gamma \in \hat{G}$ has at most one representation

(3.14) $\gamma = \pm\theta_1 \pm \cdots \pm \theta_k, \qquad \theta_i \in \Theta, \theta_i \neq \theta_j$ for $1 \leqslant i \leqslant j \leqslant k$

and the sign in (3.14) before θ_i is always chosen $+$ if θ_i has order two.

Let $\mathbf{a}: \Theta \to \mathbf{C}$ be any function such that $|\mathbf{a}(\theta)| \leqslant \frac{1}{2}$ if θ does not have order two, and $-1 \leqslant \mathbf{a}(\theta) \leqslant 1$ if θ has order two, and let Θ be dissociate. We construct the *Riesz product based on Θ and $\mathbf{a}$* as the weak* limit of the measures ν of the form $\nu = \prod q_\theta dx$, where $q_\theta = 1 + \mathbf{a}(\theta)\theta$ if θ has order two and $q_\theta = 1 + (\mathbf{a}(\theta)\theta + \mathbf{a}(\theta)^- \theta^-)$ if θ does not have order two. The limit is taken over increasing finite subsets of Θ. It is easy to show that $\hat{\mu} \in C_0(\hat{G})$ if and only if $\mathbf{a}$ vanishes at infinity on Θ. When Θ is dissociate, $\Omega(\Theta)$ will denote all elements (first order words) $\gamma \in \hat{G}$ that have representations of the form (3.14). For more about Riesz products, see [**B**], [**BM1**], [**HeZu**], [**P**], [**SZ**], [**Z**].

In what follows, we shall observe the following convention. If Θ is a dissociate set (finite or infinite) and $\mathbf{a}: \Theta \to \mathbf{C}$, then we shall assume (or

require) that $|\mathbf{a}(\theta)| \leqslant \frac{1}{2}$ if θ does not have order two and $-1 \leqslant \mathbf{a}(\theta) \leqslant 1$ otherwise.

3.9. LEMMA [**BM1**, Lemmas 1 and 4 of §3]. *Let μ, ν be Riesz products on the compact abelian group G that are based on Θ and $\mathbf{a}$ and Θ' and $\mathbf{a}'$, respectively. If*

$$\sum_{\Theta \cup \Theta'} |\mathbf{a}(\theta) - \mathbf{a}'(\theta)|^2 = \infty$$

then μ and ν are mutually singular. (We define $\mathbf{a}$ ($\mathbf{a}'$) to be zero on $\Theta' \setminus \Theta$ ($\Theta \setminus \Theta'$).) If $\lim \sup\{|\mathbf{a}(\theta) + \mathbf{a}'(\theta)|: \theta \in \Theta \cup \Theta', \theta + \theta = 0\} < 2$ and $\lim \sup\{|\mathbf{a}(\theta) + \mathbf{a}'(\theta)|: \theta \in \Theta \cup \Theta', \theta + \theta \neq 0\} < 1$, then μ and ν are mutually absolutely equivalent if (3.15) does not hold.

PROOF. This is proved in [**BM1**], [**P**]. (Note that the above definition of a Riesz product is notationally distinct from that in [**BM1**].) We will not include a proof here. In fact, we do not need the full force of 3.9. We use only the weaker statement:

3.9′. LEMMA. *Let μ be a Riesz product based on Θ and $\mathbf{a}$. Suppose that $n \geqslant 1$ and*

(3.16) $$\sum_{\theta \in \Theta} |\mathbf{a}(\theta)|^{2n} = \infty \quad and \quad \sum_{\theta \in \Theta} |\mathbf{a}(\theta)|^{2n+2} < \infty.$$

Then $\mu, \mu^2, \ldots, \mu^{n+1}$ are pairwise mutually singular and μ^{n+1} is absolutely continuous.

PROOF. This, of course, follows from 3.9, since Haar measure is a Riesz product (with $\mathbf{a} \equiv 0$). It follows as well from a famous theorem of Zygmund [**Z**, Volume I, pp. 209–211] (when $G = \mathbf{T}$) and Hewitt and Zuckerman [**HeZu**] (for more general compact abelian groups). That famous theorem is this: The Riesz product ν based on Θ and $\mathbf{a}$ is singular with respect to Haar measure if $\sum_\theta |a(\theta)|^2 = \infty$. Otherwise ν is absolutely continuous. Here is a proof of 3.9′ based on the Zygmund/Hewitt-Zuckerman result. (This argument is due to Y. Meyer [**P**].)

Suppose that μ^k and μ^j are not mutually singular for some $0 \leqslant k < j \leqslant n + 1$. Then $\mu^{(n+1)-j} * \mu^k$ and $\mu^{(n+1)-j} * \mu^j = \mu^{n+1}$ are not mutually singular. (For a proof, see, e.g., [**Sr**].) But the Zygmund/Hewitt-Zuckermann theorem says that $\mu^{n+1-j+k}$ is purely singular, while μ^{n+1} is absolutely continuous. This contradiction proves 3.9′.

For a thorough discussion of 3.9, its variants and predecessors, see [**BM1**]. A detailed discussion of the following theorem appears in Doss [**D**].

3.10. LEMMA [**D**, p. 220]. (i) *Let $\mu \in M(G)$ be singular with respect to Haar*

measure. Let $X \subseteq \hat{G}$ be compact and $\varepsilon > 0$. Then there exists a finite subset $X' \subseteq \hat{G} \backslash X$ such that $\|\hat{\mu}|_{X'}\|_{B(X')} \geqslant \|\mu\| - \varepsilon$.

(ii) *Let $\mu \in L^1(G)$ and $\varepsilon > 0$. Then there exists a compact set $X \subseteq \hat{G}$ such that $\|\hat{\mu}|_{\hat{G} \backslash X}\|_{A(\hat{G} \backslash X)} < \varepsilon$.*

3.11. LEMMA. *Let G be a compact abelian group and let $\Theta \subseteq \hat{G}$ be an infinite dissociate set. Let μ be the Riesz product based on Θ and $\mathbf{a}$, and suppose that (3.16) holds for some integer $n \geqslant 1$. Let $c_1, \ldots, c_n \in \mathbf{C}$ and $\varepsilon > 0$. Then there exists a trigonometric polynomial $f(x) = \sum_1^m d_k \langle \gamma_{k,x} \rangle$ with $\sup|f(x)| \leqslant 1$, and $\{\gamma_k\}_{k=1}^m \subseteq \Omega(\Theta)$ such that*

$$(3.17) \qquad \left| \int f(x) d\left(\sum_1^n c_j \mu^j \right) \right| \geqslant \sum_1^n |c_j| - \varepsilon.$$

PROOF. Let ν be the Riesz product generated by Θ and $\mathbf{b}(\theta) = \frac{1}{3}$, for all $\theta \in \Theta$. Then $\nu * \mu^j$ is the Riesz product generated by $\frac{1}{3}\mathbf{a}^j$. Lemma 3.9 (or the calculation of Meyer used in the proof of 3.9') shows that $\{\nu * \mu^j\}_{j=1}^{n+1}$ is a family of pairwise mutually singular measures. Let $g(x) = \sum e_r \langle r, x \rangle$ be any trigonometric polynomial with $\sup|g(x)| \leqslant 1$ and $|\int g(x) d(\sum c_j \nu * \mu^j)| \geqslant \sum |c_j| - \varepsilon$. Set $f(x) = g(x) * \nu$. Then f is a trigonometric polynomial of supremum at most one and frequencies in $\Omega(\Theta)$. A straightforward calculation shows that $\int f(x) d\mu^j = \int g(x) d(\nu * \mu^j)$ for $1 \leqslant j \leqslant n$. The lemma is proved.

3.12. COROLLARY. *Let μ be a Riesz product on the compact abelian group G, and suppose that μ satisfies (3.16), where μ is based on Θ and $\mathbf{a}$ and $n \geqslant 1$. Let $K > 0$, $\varepsilon > 0$, and $F \subseteq \hat{G}$ a finite set. Then there exists a finite set $E \subseteq \Omega(\Theta) \backslash F$ such that*

$$(3.18) \qquad |c_j| \leqslant K \, (1 \leqslant j \leqslant n) \text{ imply } \left\| \left. \sum_1^n c_j \hat{\mu}^j \right|_E \right\|_{A(E)} \geqslant \sum |c_j| - \varepsilon.$$

PROOF. It is easy to see that we can take any finite set Φ in Θ, redefine $\mathbf{a}$ to be zero there, obtain a new Riesz product, and (3.16) will still hold. Furthermore, the finite subset of Θ may be chosen so as to have $F \cap \Omega(\Theta) \backslash \Phi = \varnothing$. (This follows from the uniqueness of the expressions (3.14).)

3.13. LEMMA. *Let G be a compact abelian group, and let ν be a Riesz product on G that is based on Θ and $\mathbf{a}$. Suppose that ν satisfies (3.5)–(3.6), and that $n \geqslant 1$. Then there exists an infinite sequence of finite subsets Φ_1, Φ_2, $\ldots$ of Θ that are pairwise disjoint and such that*

$$(3.19) \qquad \sum_{j=1}^\infty \left(\prod_{\theta \in \Phi_j} |\mathbf{a}(\theta)| \right)^{2n} = \infty \quad \text{and} \quad \sum_{j=1}^\infty \left(\prod_{\theta \in \Phi_j} |\mathbf{a}(\theta)| \right)^{2n+2} < \infty.$$

REMARK. It is easy to see that the set $\Psi = \{\phi_j\}_{j=1}^{\infty}$, where $\varphi_j = \sum_{\theta \in \Phi_j} \theta$ is dissociate, and that the restriction of $\hat{\nu}$ to $\Omega(\Psi)$ agrees with $\hat{\mu}$, the transform of the Riesz product based on Ψ and $\mathbf{b}$, where $\mathbf{b}(\varphi_j) = \prod_{\theta \in \Phi_j} \mathbf{a}(\theta)$. Hence, (3.19) is a "subordinate" version of (3.16).

PROOF OF 3.13. If $\lim \sup |\mathbf{a}(\theta)| = 0$, then it is easy to find singletons Φ_j that have the required properties. If $\lim \sup |\mathbf{a}(\theta)| = a \neq 0$, then a simple induction will produce the sets Φ_j (whose cardinalities will be increasing, in general). The details are left to the reader.

3.14. PROOF OF THEOREM 3.2. (i) We fix $n \geqslant 1$, $K > 0$, and $\varepsilon > 0$. By 3.13 and the remark following 3.13, there exists a Riesz product, μ, based on Ψ and $\mathbf{b}$, such that (3.16) holds for μ and such that $\hat{\mu}(\omega) = \hat{\nu}(\omega)$ for all $\omega \in \Omega(\Psi)$. Since $\mu^{n+1} \in L^1(G)$, we may apply 3.10(ii), and find a finite subset $F \subseteq \hat{G}$ such that $\|\hat{\mu}^{n+1}|_{G \setminus F}\|_{A(G \setminus F)} < \varepsilon$. An application of 3.12 completes the proof.

(ii) The argument is similar. We need a new version of 3.12, but it is apparent what the statement is, and how the proof of the present 3.12 may be modified to yield the new result. We then proceed as in the first part of the theorem.

4. Some questions.

4.1. Classify all measures with property P (or $\tilde{P}$). We used an l^2 criterion in §2. Do all symmetric independent power probability measures have property P? Do Bernoulli convolutions? See [**BM2**].

4.2. Which homogeneous polynomials $P_k(z)$ of degree k operate in $PD_1(\hat{G})$ for exceptional groups $\hat{G}$? See [**Ri1**].

4.3. Suppose μ is a Riesz product and $F: \{z: z \in \hat{\mu}(\hat{G})\} \to \mathbf{C}$ operates on $\hat{\mu}$. When does F agree with an analytic function on $\hat{\mu}(\hat{G})$? See §3 above.

4.4. It is not hard to show that if G contains $\mathbf{T}$, $\mathbf{R}$ or $\mathbf{Z}$ as closed subgroups and G is not compact and $F: D(G) \to \mathbf{C}$ operates in $PD_1(\hat{G})$, then F is continuous. When $G = \mathbf{Z}(p^\infty)$, discontinuous functions operate in $PD_1(\hat{G})$. What is the general situation?

BIBLIOGRAPHY

BBM. W. J. Bailey, G. Brown and W. Moran, *Spectra of independent power measures*, Proc. Cambridge Philos. Soc. **72** (1972), 27–35. MR **45** #5667.

Bl. R. C. Blei, *On trigonometric series associated with separable, translation-invariant subspaces of $L^\infty(G)$*, Trans. Amer. Math. Soc. **173** (1973), 491–499. MR **47** #2269.

B. G. Brown, *Riesz products and generalized characters*, Proc. London Math. Soc. (3) **30** (1975), 209–238. MR **51** #8737.

BM1. G. Brown and W. Moran, *On orthogonality of Riesz products*, Proc. Cambridge Philos. Soc. **76** (1974), 173–181. MR **50** #2812.

BM2. ———, *Coin tossing and powers of singular measures*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 349–364. MR **50** #11363.

D. R. Doss, *Approximations and representations for Fourier transforms*, Trans. Amer. Math. Soc. **153** (1971), 211–221. MR **42** #3494.

Dr. S. W. Drury, *Sur les ensembles de Sidon*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A162–A163. MR **42** #6530.

Gr1. C. C. Graham, *Symbolic calculus for positive-definite functions*, J. Functional Analysis **11** (1972), 465–478.

Gr2. ———, *The functions which operate on Fourier-Stieltjes transforms of L-subalgebras of M(T)*, J. Functional Analysis **11** (1972), 453–464.

Gr3. ———, *A Riesz product proof of the Wiener-Pitt theorem*, Proc. Amer. Math. Soc. **44** (1974), 312–314. MR **49** #5722.

Gr4. ———, $M_0(G)$ *is not a prime L-ideal of measures*, Proc. Amer. Math. Soc. **27** (1971), 557–562. MR **43** #3739.

Hz. C. S. Herz, *Fonctions opérant sur les fonctions définies-positives*, Ann. Inst. Fourier (Grenoble) **13** (1963), fasc. 1, 161–180. MR **27** #2806; **30**, p. 1204.

HR. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. I, II, Academic Press, New York, 1963; Springer-Verlag, Berlin and New York, 1970. MR **28** #158; **41** #7378; erratum, MR **42**, p. 1825.

HeZu. E. Hewitt and H. S. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Cambridge Philos. Soc. **62** (1966), 399–420; *Corrigendum*, ibid. **63** (1967), 367–368. MR **33** #1655; **34** #8097.

Ho. L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N.J., 1966. MR **34** #2933.

Kz. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968. MR **40** #1734.

Kf1. R. Kaufman, *On the symbolic calculus of Fourier transforms*, Israel J. Math. **6** (1968), 30–35.

Kf2. ———, *An example in the calculus of Fourier transforms*, Bull. Amer. Math. Soc. **73** (1967), 357–359. MR **35** #3373.

Ku. A. G. Kuroš, *The theory of groups*, 2nd ed., Gosudarstv., Moscow, 1953; English transl., Chelsea, New York, 1955. MR **15**, 501; **17**, 124.

Ma. D. K. Mason, *The functions which operate on Fourier-Stieltjes transforms of certain subvector spaces of M(T)*, Thesis, Northwestern Univ., Evanston, Ill., 1972.

———, *The functions which operate on Fourier-Stieltjes transforms of certain subvector spaces of M(T)*, Notices Amer. Math. Soc. **20** (1973), A-389. Abstract #704-B13.

Mo. W. Moran, *The individual symbolic calculus for measures*, Proc. London Math. Soc. (3) **31** (1975), 385–417. MR **52** #8790.

P. J. Peyrière, *Sur les produits de Riesz*, C. R. Acad. Sci., Paris Sér. A-B **276** (1973), A1417–A1419. MR **47** #5512.

Ri1. D. Rider, *Functions which operate on positive-definite functions*, Proc. Cambridge Philos. Soc. **69** (1971), 87–97.

Ri2. ———, *Functions which operate on $\mathcal{F}L_p(T)$, $1 < p < 2$*, Pacific J. Math. **40** (1972), 681–693. MR **46** #5926.

Ri3. ———, *Functions which operate in the Fourier algebra of a compact group*, Proc. Amer. Math. Soc. **28** (1971), 525–530. MR **43** #2532.

Ri4. ———, *Gap series on groups and spheres*, Canad. J. Math. **18** (1966), 389–398. MR **32** #8047.

Ru1. W. Rudin, *Fourier analysis on groups*, Wiley, New York, 1962. MR **27** #2808.

Ru2. ———, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227. MR **22** #6972.

S. S. Saeki, *Operating functions on $B_0(\hat{G})$ in plane regions*, Tôhoku Math. J. (2) **21** (1969), 112–116. MR **40** #647.

SZ. R. Salem and A. Zygmund, *On a theorem of Banach*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 293–295. MR **9**, 88.

SV. D. L. Salinger and N. Th. Varopoulos, *Convolutions of measures and sets of analyticity*, Math. Scand. **25** (1969), 5–18. MR **41** #8930.

S. Yu. A. Šreĭder, *The structure of maximal ideals in rings of measures with convolution*, Mat.

Sb. (N.S.) **27 (69)** (1950), 297–318; English transl., Amer. Math. Soc. Transl. **81** (1953). MR **12**, 240; **14**, 768.

T1. J. L. Taylor, *Measure algebras*, CBMS Regional Conf. Ser. in Math., no. 16, Amer. Math. Soc., Providence, R.I., 1973.

T2. ——, *On the spectrum of a measure*, Advances in Math. **12** (1974), 451–463. MR **50** #10687.

V1. N. Th. Varopoulos, *The functions that operate on $B_0(\Gamma)$ of a discrete group Γ*, Bull. Soc. Math. France **93** (1965), 301–321. MR **34** #565.

V2. ——, *Measure algebras of a locally compact abelian group*, Séminaire Bourbaki, Vol. 1964/1965, Exposé No. 282, Benjamin, New York and Amsterdam, 1966. MR **33** #54201.

V3. ——, *Tensor algebras and harmonic analysis*, Acta Math. **119** (1967), 51–112. MR **39** #1911.

V4. ——, *Tensor algebras over discrete spaces*, J. Functional Analysis **3** (1969), 321–335. MR **40** #3328.

Z. A. Zygmund, *Trigonometric series*. Vols. I, II, 2nd ed. reprinted with corrections and additions, Cambridge Univ. Press, London and New York, 1968. MR **38** #4882.

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